

Supersymmetry and Lie groups

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Abstract

We construct all vacuum states of $\mathcal{N} = 2$ supersymmetric Yang-Mills quantum mechanics (for $SU(N)$ group) and discuss their origin from the $SU(N)$ real cohomology.

1 Introduction

Supersymmetric quantum mechanics (SQM) provides an elegant and deep connection between geometry and physics [1]. It was observed in early eighties by Witten that the number of the vacuum states of $\mathcal{N} = 2$ nonlinear sigma model on manifold M coincides with the Betti numbers of the manifold. In this paper we find an analogous correspondence in seemingly different models namely $\mathcal{N} = 2$ supersymmetric Yang-Mills quantum mechanics (SYMQM) with $SU(N)$ gauge group. It

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turns out that the number of vacuum states in sector with n_F fermions coincides with the n_F 'th Betti number of $SU(N)$ manifold. Such relations to the topology, while well understood in the nonlinear sigma model, were not, to our knowledge fully exhibited for this system.

In the following section, after determining all vacuum states of $\mathcal{N} = 2$ SYMQM, we identify them with nontrivial cocycles on $SU(N)$. This serves as a mathematical proof that the number of vacuum states coincides with the $SU(N)$ Betti numbers. We then conclude that one can perform an identification, à la Witten, between the left invariant forms on a Lie group and the fermion matrices $\bar{\psi} = T_a \bar{\psi}_a$ where T_a are matrix group generators and $\bar{\psi}_a$ are fermion fields.

2 A setup

The $\mathcal{N} = 2$ SYMQM involves real scalar ϕ_a , real gauge potential A_a and the complex fermion field $\psi_a, \bar{\psi}_a$ all in the adjoint representation of $SU(N)$. The system can be obtained by the dimensional reduction from $D = 1+1, \mathcal{N} = 1$ gauge theory by rewriting the potential $A_a{}_\mu, \mu = 0, 1$ as $A_a{}_\mu = (A_a{}_0, A_a{}_1) \equiv (A_a, \phi_a)$. The resulting lagrangian is [2]

$$L = \frac{1}{2}(D_t\phi)_a(D_t\phi)_a + i\bar{\psi}_a(D_t\psi)_a - igf_{abc}\bar{\psi}_a\phi_b\psi_c,$$

where covariant derivatives are

$$(D_t\phi)_a = \partial_t\phi_a - gf_{abc}A_b\phi_c, \quad (D_t\psi)_a = \partial_t\psi_a - gf_{abc}A_b\psi_c.$$

The quantization of the system gives the hamiltonian

$$H = \frac{1}{2}p_ap_a + g\phi_aG_a, \quad G_a = if_{abc}(\phi_bp_c + \psi_b\bar{\psi}_c), \quad [G_a, G_b] = if_{abc}G_c,$$

where ϕ_a, p_b and $\psi_a, \bar{\psi}_b$ are conjugate variables $[\phi_a, p_b] = i\delta_{ab}$, $\{\psi_a, \bar{\psi}_b\} = i\delta_{ab}$. The Gauss law after dimensional reduction becomes the singlet constraint on physical states $|s\rangle$, $G_a |s\rangle = 0$. It follows now that in the subspace of $SU(N)$ singlets the hamiltonian is very simple

$$H = \frac{1}{2}p_a p_a = \frac{1}{2}\{Q, \bar{Q}\}, \quad Q^2 = \bar{Q}^2 = 0, \quad Q = \psi_a p_a, \quad \bar{Q} = \bar{\psi}_a p_a,$$

with the supercharges Q, \bar{Q} . Therefore the model is completely described by the following equations

$$\frac{1}{2}p_a p_a |s\rangle = E |s\rangle, \quad G_a |s\rangle = 0.$$

To make the system well defined we still have to take care of the normalization of the vacuum $|v\rangle$: $H |v\rangle = 0$ in the purely bosonic sector $\psi |v\rangle = 0$ (it is worth emphasizing that the vacuum state of this model differs from the Fock vacuum $|0\rangle$: $\psi_a |0\rangle = (\phi_a + ip_a) |0\rangle = 0$). Unfortunately if ϕ_a 's are noncompact then this requirement is not satisfied because the wave function of the vacuum $|v\rangle$ in coordinate representation is simply the constant function, $\langle\phi | v\rangle = 1$. In fact the situation is even worse since one can prove [3] that there exist an infinite number of polynomials $P_k(Tr(\phi^2), \dots, Tr(\phi^N))$ such that

$$[p_a p_a, P_k] = 2ik\partial_a P_k p_a \quad \partial_a = \frac{\partial}{\partial\phi_a}.$$

where we used the notation

$$Tr(AB\dots) = A_a B_b \dots Tr(T_a T_b \dots),$$

where T_a 's are $SU(N)$ generators in the fundamental representation satisfying

$$T_a T_b = \frac{2}{N}\delta_{ab} + (d_{abc} + if_{abc})T_c = \frac{2}{N}\delta_{ab} + \frac{1}{2}Tr(T_a T_b T_c)T_c.$$

where d_{abc}, f_{abc} are $SU(N)$ structure tensors.

Therefore, if $|v\rangle$ is the vacuum state then so is $P_k |v\rangle$ hence there are infinitely many vacua in purely bosonic sector. Moreover, since the vacuum state is not normalizable the basic theorem in supersymmetry namely

$$Q |v\rangle = 0, \bar{Q} |v\rangle = 0 \iff H |v\rangle = 0,$$

does not have to hold anymore and it doesn't. To see this explicitly we take the polynomial $Tr(\phi^3)$. We have

$$p_a p_a Tr(\phi^3) |v\rangle = 0,$$

but

$$\bar{Q} Tr(\phi^3) |v\rangle = -3i Tr(\phi^2 \bar{\psi}) |v\rangle \neq 0.$$

To remedy this ill situation we compactify the coordinates $\phi_a \in [0, 1]$ and impose the periodicity condition on the wave function $\Psi(\phi_a) = \Psi(\phi_a + 1)$ in all fermion sectors. The lagrangian is not invariant under the shift $\delta\phi_a = 1$ but we also have $\delta L = -ig\phi_a G_a$ therefore in the space of physical states the compactification is properly imposed. The condition $\Psi(\phi_a) = \Psi(\phi_a + 1)$ furnishes out all additional solutions $P_k |v\rangle$ since they are not periodic in ϕ_a . Therefore there is only one vacuum $|v\rangle$, in the sector with no fermions, which is now normalizable.

3 The number of vacuum states

Here we compute the number of vacua, of the model described in previous section, in sectors with fermions. An arbitrary state with k fermions can be written as

$$t_{i_1 \dots i_k}(\phi) \bar{\psi}_{i_1} \dots \bar{\psi}_{i_k} |v\rangle,$$

where $t_{i_1 \dots i_k}(\phi)$ are some functions depending on ϕ_a . One can proceed in two independent ways to count the vacua a) by explicitly constructing the states and b) with use of the representation theory.

We start with the first approach. The general form of the vacuum state in the sector with k fermions is

$$|v\rangle_k = t_{i_1 \dots i_k} \bar{\psi}_{i_1} \dots \bar{\psi}_{i_k} |v\rangle, \quad (1)$$

where $t_{i_1 \dots i_k}$ is $SU(N)$ invariant (due to the singlet constraint) tensor. There are no ϕ_a s in (1) since any appearance of them gives $\bar{Q} |v\rangle_k \neq 0$. By the same reason we act in (1) with fermion operators on the bosonic vacuum $|v\rangle$ rather than on the Fock vacuum $|0\rangle$ since the wave function corresponding to $|0\rangle$ depends on ϕ_a i.e. $\langle \phi | 0 \rangle \propto \exp(-Tr(\phi^2)/2)$.

Invariant tensors can be expressed as linear combination of products of trace tensors $Tr(T_a T_b \dots)$ therefore the following states

$$Tr(\bar{\psi}^2)^{i_2} \dots Tr(\bar{\psi}^{N^2-1})^{i_{N^2-1}} |v\rangle,$$

span the entire space of vacuum states in all fermion sectors. Moreover, since fermions anticommute we have $Tr(\bar{\psi}^{2k}) = 0$ and $Tr(\bar{\psi}^{2k+1})^2 = 0$. Therefore we are left with the states

$$Tr(\bar{\psi}^3)^{i_3} Tr(\bar{\psi}^5)^{i_5} \dots Tr(\bar{\psi}^{N^2-1})^{i_{N^2-1}} |v\rangle, \quad i_k = 0, 1, \quad N^2 - 1 \text{ odd},$$

$$Tr(\bar{\psi}^3)^{i_3} Tr(\bar{\psi}^5)^{i_5} \dots Tr(\bar{\psi}^{N^2-2})^{i_{N^2-2}} |v\rangle, \quad i_k = 0, 1, \quad N^2 - 2 \text{ odd}.$$

They can be further reduced due to the following fact. The multiplication law for T_a 's gives us

$$\bar{\psi} \bar{\psi} = \frac{1}{2} Tr(\bar{\psi} \bar{\psi} T_a) T_a, \quad \bar{\psi} = \bar{\psi}_a T_a,$$

therefore

$$Tr(\bar{\psi}^{2n+1}) = \frac{1}{2^n} Tr(\bar{\psi}\bar{\psi}T_{a_1}) \dots Tr(\bar{\psi}\bar{\psi}T_{a_n}) Tr(T_{a_1} \dots T_{a_n} \bar{\psi}).$$

Since operators $Tr(\bar{\psi}\bar{\psi}T_{a_k})$ commute with each other we may symmetrize over indices

$$Tr(\bar{\psi}^{2n+1}) = \frac{1}{2^n n!} Tr(\bar{\psi}\bar{\psi}T_{a_1}) \dots Tr(\bar{\psi}\bar{\psi}T_{a_n}) Tr(T_{a_1} \dots T_{a_n} \bar{\psi}).$$

Generators T_a are $N \times N$ matrices therefore according to Cayley-Hamilton theorem if $n \geq N$ then the matrix $T_{a_1} \dots T_{a_n}$ can be expressed as a linear combination of products of matrices $T_{a_1} \dots T_{a_k}$, $k < n$. This implies that operators $Tr(\bar{\psi}^{2n+1})$, $n \geq N$ can be expressed as a linear combination of products of operators $Tr(\bar{\psi}^{2n+1})$, $n < N$.

Therefore we are left with the following vacuum states

$$|v\rangle_k = Tr(\bar{\psi}^3)^{i_3} Tr(\bar{\psi}^5)^{i_5} \dots Tr(\bar{\psi}^{2N-1})^{i_{2N-1}} |v\rangle, \quad i_k = 0, 1.$$

Let us denote the number of vacua in sector with k fermions by b_k . It follows that the generating polynomial for b_k 's is

$$P(t) = \sum_{i=0}^{N^2-1} b_i t^i = (1+t^3)(1+t^5) \dots (1+t^{2N-1}), \quad (2)$$

which is exactly the Poincaré polynomial for the $SU(N)$ manifold (the collection of Poincaré polynomials for other compact semisimple Lie groups can be found in, e.g. [4] or [5]). This result is somewhat puzzling since it is not entirely clear why the $\mathcal{N} = 2$, SYMQM should have any topological interpretation analogous to nonlinear sigma models. Before we give the answer to this puzzle we will present yet another derivation of the above result with use of representation theory.

Let F_a be the vector space spanned by operators $\bar{\psi}_a$. The fermions are in the adjoint representation of $SU(N)$ which we denote by R . It follows that the state with k fermions belongs to the tensor product $V = Alt(\otimes_{a=1}^k F_a)$ where Alt means the antisymmetrization of the tensor product. The number of independent vacua b_k is simply the number of $SU(N)$ singlets in V therefore

$$b_k = \int d\mu_{\mathbf{SU}(N)} \chi_{Alt}^{[k]}(R),$$

where the $SU(N)$ invariant measure $d\mu_{\mathbf{SU}(N)}$, the antisymmetric power of R , $\chi_{Alt}^{[a_F]}(R)$ and the characters $\chi(R)$ are listed in the Appendix where we also prove that the generating function (2) has the following integral representation

$$P(t) = \frac{1}{N!} (1-t)^{N-1} \int_{[0,2\pi]^N} \prod_i \frac{d\alpha_i}{2\pi} \delta(\alpha_N) \prod_{i \neq j} (1 - \frac{z_i}{z_j}) (1 - t \frac{z_i}{z_j}), \quad (3)$$

where $z_j = e^{i\alpha_j}$ and $\alpha_j = [0, 2\pi]$. For given N the above integral can be evaluated in terms of residues and it reproduces (2) as it should.

The connection with the group theory becomes even more evident if we realize that the vacuum states correspond to the non-trivial cocycles for the $SU(N)$ real cohomology. To be more specific, (see e.g. [5] for more details), consider a basis $X_1|_e, \dots, X_{N^2-1}|_e$ of tangent space $T_e SU(N)$, where e is the identity element of $SU(N)$. One can define the basis of linearly independent left-invariant vector fields X_1, \dots, X_{N^2-1} at each point $g \in SU(N)$ by $X_a|_g = L_{g*} X_a|_e$, where L_{g*} is the $SU(N)$ automorphism induced by the left translation L_g . Let θ^a be dual to $X_a|_g$ and let us consider the Maurer-Cartan form $\theta(g) = \theta^a(g) X_a$. Due to the

Maurer-Cartan equation $d\theta = -\theta \wedge \theta$ the following form

$$\Omega^n(g) = Tr(\underbrace{\theta \wedge \dots \wedge \theta}_n), \quad n - \text{odd},$$

is closed but not exact, therefore it defines the well known Chevalley-Eilenberg n -cocycle of $SU(N)$. We recognize that the cocycles $\Omega^n(g)$ correspond to operators $Tr(\bar{\psi}^n)$ needed to construct the vacuum states¹. Therefore, the number of vacuum states in the sector with n_F fermions coincides with the number of independent, closed, not exact, cocycles that one can build on $SU(N)$, i.e. the n_F 'th Betti number.

One can continue the analogy between $Tr(\bar{\psi}^n)$ and $\Omega^n(g)$ by identifying $\bar{\psi}$ with the Maurer-Cartan 1-form θ and the multiplication of fermions $\bar{\psi}\bar{\psi}$ with the wedge product $\theta \wedge \theta$. The vacuum state in the bosonic sector is then a constant 0-form equal 1.

4 Summary

In this paper we investigated the vacuum structure of $\mathcal{N} = 2$, SYMQM. It turns out that the vacua reveal a topological information of the gauge group considered. If we look at this system just regarding the hamiltonian and the singlet constraint it is unclear why there should be any such information. The necessity of the singlet constraint can be seen from the following argument. One could consider the hamiltonian $H = \frac{1}{2}p_a p_a$ without the constraint and the system still remains

¹ The explicit construction of coordinates of $\Omega^n(g)$ and their properties are discussed in [6].

supersymmetric only this time the number of vacuum states does not coincide with Betti numbers.²

It turns out that the construction of vacuum states coincides with the well known construction of non-trivial cocycles on $SU(N)$. The coordinates of those cocycles are group invariant tensors which is precisely the requirement coming from the Gauss law.

We focused entirely on the $SU(N)$ group, however the case of other Lie groups is analogous although it is necessary for the Lie group to be compact.

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²This time the number of vacuum states and the corresponding generating polynomials are

$$b_{n_F} = \binom{N^2 - 1}{n_F}, \quad P(t) = (1 + t)^{N^2 - 1}.$$

6 Appendix A

Here we give the conventions used in section 3 and prove Eqn. (2). The conventions we use can be found in [7]. The $SU(N)$ invariant, normalized, measure is

$$d\mu_{SU(N)} = \frac{1}{N!} \prod_{i=1}^N \frac{d\alpha_i}{2\pi} \delta_P \left(\sum_{i=1}^N \alpha_i \mid M \mid^2, \quad \alpha_i \in [0, 2\pi], \right)$$

where δ_P is a periodic Dirac delta with period 2π

$$\delta_P(x) = \sum_k \delta(x - 2\pi k),$$

the measure factor M is given by Vandermonde determinant

$$M = \text{Det}(z_j^{(N-i)}) = \prod_{i < j} (z_i - z_j), \quad z_j = e^{i\alpha_j},$$

and $\chi_{Alt}^{[a_F]}(R)$ is the antisymmetric power of R given by Frobenius formula

$$\chi_{Alt}^{[a_F]}(R) = \sum_{\sum_k k i_k = n_F} (-1)^{\sum_k i_k} \prod_{k=1}^{n_B} \frac{1}{i_k!} \frac{\chi^{i_k}(R^k)}{k^{i_k}}, \quad (4)$$

where the characters χ are given by Weyl determinant formula

$$\chi(R) \equiv \chi(\{\alpha_i\}_{i=1}^N) = \frac{\text{Det}(z_j^{(N-i+l_i)})}{\text{Det}(z_j^{(N-i)})}, \quad \chi(R^k) = \chi(\{k\alpha_i\}_{i=1}^N).$$

The numbers l_i enumerate the representation in which the character is computed.

In our case it is the adjoint representation of $SU(N)$ therefore $(l_1, l_2, \dots, l_N) = (2, 1, \dots, 1, 0)$. In this representation the characters simplify into

$$\chi_{SU(N)}(\{\alpha_i\}) = \sum_{i,j} \frac{z_i}{z_j} - 1. \quad (5)$$

Since for $SU(N)$ we have $\chi_{Alt}(R^k) = 0$ when $k > N^2 - 1$ we can write the generating function (2) as an infinite sum

$$P(t) = \sum_{i=0}^{\infty} b_i t^i. \quad (6)$$

Substituting (3), (4), (5) to (6) we obtain (after some manipulations)

$$P(t) = \int d\mu_{\mathbf{SU}(\mathbf{N})} \exp \left(\sum_{k=1}^{\infty} \frac{t^k}{k} \chi(\{k\alpha_i\}_{i=1}^N) \right).$$

Using the formula for characters and the measure we obtain

$$P(t) = \frac{1}{N!} \int_{[0,2\pi]^{N-1}} \delta(\alpha_N) \prod_i \frac{d\alpha_i}{2\pi} \prod_{i \neq j} \left(1 - \frac{z_i}{z_j}\right) \prod_{i,j} \left(1 - t \frac{z_i}{z_j}\right),$$

where we also changed variables $z_i \rightarrow z_i / \prod_{j=1}^N z_j$, $z_N \rightarrow \prod_{j=1}^N z_j$.

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